TUPT-03-99
april 1999

Relic Gravitons, Dominant Energy Condition and Bulk Viscous Stresses

Massimo Giovannini [1]

Institute of Cosmology, Department of Physics and Astronomy
Tufts University, Medford, Massachusetts 02155, USA

Abstract

If the energy momentum tensor contains bulk viscous stresses violating the dominant energy condition (DOC) the energy spectra of the relic gravitons (produced at the time of the DOC's violation) increase in frequency in a calculable way. In a general relativistic context we give examples where the DOC is only violated for a limited amount of time after which the ordinary (radiation dominated) evolution takes place. We connect our discussion to some recent remarks of Grishchuk concerning the detectability of the stochastic gravitational wave background by the forthcoming interferometric detectors.

[1]Electronic address: giovan@cosmos2.phy.tufts.edu

Every transition of the background geometry leads, inevitably, to the production of stochastically distributed relic gravitons whose energy spectra represent a crucial probe of the very early stages of the evolution of our Universe [1]. Prior to the nucleosynthesis epoch there are no direct test of the thermodynamical state of the Universe and the presence of an inflationary phase of expansion is usually justified by causality arguments applied to the Cosmic Microwave Background (CMB) photons whose emission regions could not have been in causal contact in the far past if a never ending radiation dominated phase would precede our present matter dominated stage of expansion [2].

An inflationary evolution, if regarded at an effective level, violates the (general relativistic) strong energy condition (SEC) namely $\rho + 3p \geq 0$ where ρ and p are, respectively, the energy density and the pressure density of the perfect fluid sources driving the evolution of the background geometry [3]. The need for such a violation can be immediately seen from the structure of Friedmann equations which imply that, if $\ddot{a} > 0$, the SEC needs to be violated ($a(t)$ is the scale factor of the homogeneous and isotropic Universe and the over-dot denotes the differentiation with respect to the cosmic time t).

Recently, Grishchuk [4, 5] made an interesting observation concerning the slopes of the energy spectra of relic gravitons emerging from the models of the early Universe. In short the argument goes as follows. Suppose that we are in the framework of general relativity and suppose, as it is usually done, that the effective sources driving the background geometry can be parameterized by a stress tensor with perfect fluid form. As we stressed at the beginning, there are little doubts that the Universe had to be dominated by radiation (at least) since nucleosynthesis. Prior to that epoch we do not know which kind of energy momentum tensor could approximate the background sources but we would like to deal with ever expanding Universes. Moreover, at least for some time, we would like an accelerated expansion in order to solve the so called kinematical problems of the standard cosmological model. The second of the two previous requirements necessarily leads to the violation of the SEC.

A very naive model of the early Universe would then be given by two phases. An unknown phase where the perfect fluid sources have a generic equation of state $p = \gamma\rho$, followed, at some transition time t_1, by a radiation dominated phase with $p = \rho/3$. The question we are very interested to ask is under which conditions such a toy model would produce energy spectra of relic gravitons increasing faster than the first power of the frequency. Such a question is of obvious experimental relevance since, in the near future, various interferometric detectors of gravitational waves will come in operation. Now, if the energy spectra of relic gravitons are flat (or slowly increasing) with frequency there are little hopes of detecting them. In fact, the COBE limit applied to a frequency $\nu_0 \sim 10^{-18}\ h_0$ Hz imposes that the relic gravitons energy density (in critical units) $h_0^2 \Omega_{\rm GW}$ has to be smaller then (or of the order of) 6.9×10^{-11} (h_0 is the present indetermination in the value of the Hubble constant). Due to the transition from radiation to matter, the infra-red branch of the graviton spectrum declines (in frequency) as ν^{-2} between ν_0 and $\nu_2 \sim 10^{-16}\ \Omega_0\ h_0^2$ Hz. If we take into account that the typical frequency of operation of the interferometers is between 10 and 100 Hz,

we see that it is quite interesting to understand which kind of models could give energy densities (around 100 Hz) larger than the typical inflationary models whose prediction is $h_0^2\Omega_{\rm GW} \leq 10^{-14}$.

If we confine our attention to expanding Universes (i.e. $\dot{a} > 0$) the slopes of the energy spectra are crucially related to the sign of $p+\rho$. This point can be easily seen by looking at the Friedmann equations whose solution gives $a(t) = t^{\alpha}$ with $\alpha = 2/[3(\gamma+1)]$, having assumed $p = \gamma\rho$. The spectra of relic gravitons produced because of transition of the geometry from a generic perfect fluid stage to a radiation stage can be easily computed by matching the solutions of Eq. (11) in the two temporal regions defined by the transition time t_1. The requirement that we want expanding (and inflationary) Universes implies that the energy density of the created gravitons cannot increase with frequency if $p+\rho \geq 0$.

It was noticed by Barrow [6] that a violation of the DOC does not necessarily forbid the viability of a given cosmological model in the same way as the violation of the SEC does not forbid the viability of ordinary inflationary models [8]. A violation of the DOC implies that $\dot{H} > 0$ since, by combining the Friedmann equations we get that

$$M_P^2 \dot{H} = -\frac{3}{2}(\rho + p). \tag{1}$$

Notice, moreover, that the requirement of eternal expansion implies, together with $\dot{H} > 0$, that $\ddot{a} > 0$. In the context of general relativity, cosmological solutions violating the DOC (but not violating the weak energy conditions) have been sometimes connected with the presence of bulk viscous stresses driving the evolution of the geometry.

Bulk viscous stresses [9] can appear quite naturally in cosmology as the result of the processes of particles production [10] in curved space-time . Moreover, bulk viscosity represents an interesting extension of the standard cosmological scenarios since it introduces dissipation in the model without breaking isotropy but by only redefining the space-space components of the energy momentum tensor according to

$$p' = p - 3\xi H, \tag{2}$$

where ξ is the bulk viscosity coefficient. The Friedmann equations in the presence of bulk viscous stresses can be easily written as

$$\begin{aligned} &M_P^2 H^2 = \rho, \\ &M_P^2(\dot{H} + H^2) = -\frac{1}{2}(\rho + 3p'), \\ &\dot{\rho} + 3H(\rho + p') = 0. \end{aligned} \tag{3}$$

As an example of the possibility of violating the DOC in not unreasonable models let us assume the simplest possibility namely $\xi = \text{const} > 0$ and $\gamma = -1$ which represents a particular case of the solution described by Barrow. The scale factor associated with this

solution has a typical double exponential form [6]

$$a(t) = a_0 \exp\left[\exp\left(\frac{9}{2}\xi t\right)\right] \tag{4}$$

and it describes an accelerated expansion with $\dot{H} > 0$ (we took $M_P = 1$). In the context of [6] it was argued that this behavior may be connected with a violation of the second principle of thermodynamics generalized to curved backgrounds (of quasi-de Sitter and non de Sitter type [7]). Other solutions of the type given by Eq. (4) can be constructed by assuming, for instance, $\xi = \alpha\rho^m$, m being an arbitrary power and α a proportionality constant.

A distinctive feature of the class of solutions by Eq. (4) is that they describe a violation of the DOC which occurs all the time. Namely, we can show that $\dot{H} > 0$ holds over all the times where the solution is defined.

Solutions can be constructed where the violation of the DOC is only realized for a finite amount of time. The phase leading to a violation of the DOC can then be smoothly connected with an ordinary evolution preserving both the DOC and the SEC and whose Hubble parameter declines in time. In order to have this behavior dynamically realized we have to require that the bulk viscosity coefficient will change sign at the moment connected with the transition from the phase violating the DOC to the phase where the DOC is restored. Let us then assume that $\xi = q\dot{H}/H$ where q is a constant to be determined by consistency with the Friedmann equations. The reason for this parameterization is indeed simple. As we stressed, the amount of violation of the DOC is indeed proportional to $\dot{H}$. Moreover, the compatibility with the Eqs. (3) requires that $q = (2/9)M_P^2$.

Under this ansatz, a class of solutions of Eqs. (3) is given by the scale factor

$$a(t) = \exp\left[\delta \ \mathrm{arcsinh}(t)\right] \equiv \left(t + \sqrt{t^2 + t_1^2}\right)^{\delta}, \tag{5}$$

whose associated Hubble parameter is given by

$$H = \frac{\delta}{\sqrt{t^2 + t_1^2}}. \tag{6}$$

We can immediately see by taking the limit of $a(t)$ for $t \to \pm\infty$ we get that

$$a_{\pm\infty}(t) \to (\pm t)^{\pm\delta} \tag{7}$$

In this model the explicit evolution of the shear parameter and of the energy density can be obtained, according to Eqs. (3), as

$$\xi(t) = -\frac{2}{9}M_P^2 \frac{t}{t^2 + t_1^2}, \quad \rho(t) = \frac{M_P^2\ \delta^2}{t^2 + t_1^2}. \tag{8}$$

Notice that $\xi(t)$ is positive for $t \to -\infty$ and it is negative for $t \to +\infty$. Therefore, the DOC is violated in the cosmic time interval $]-\infty, 0]$ and it is restored for $[0, +\infty[$. The energy

density and the curvature are regular and well defined for every t and, moreover, $\ddot{a} > 0$ for $]-\infty, 0]$ whereas $\ddot{a} < 0$ for $[0, +\infty[$.

If we want the phase where the DOC is violated to be smoothly connected to a radiation dominated evolution we are led to choose $\delta = 1/2$. With this choice we can see from the exact expressions of the scale factor that

$$a(t) = \left(t + \sqrt{t^2 + t_1^2}\right)^{\frac{1}{2}}, \tag{9}$$

Notice that, in this case, the pressure density goes as $p \to \rho/3$ for $t \to +\infty$. The examples we just discussed suggest a possible speculation. The initial phase where the DOC and the SEC are both violated could evolve towards a phase where the SEC and the DOC are both restored.

If this is the case we can show that the obtained graviton spectra excited by this type of background might be of some phenomenological relevance. Suppose indeed that the Universe evolves through two different stages of expansion. In the first stage the DOC and the SEC will be both violated (i.e. $a(t) \sim (-t)^{-1/2}$) In the second phase the DOC and the SEC are both restored (i.e. $a(t) \sim t^{1/2}$). If the transition from the regime $p + \rho < 0$ to the regime $p + \rho > 0$ occurs smoothly we can also expect the presence of a transition epoch where $p + \rho = 0$ and the DOC is saturated but not violated. Then by ensuring the continuity of the scale factors (and of their first derivatives) between the three regimes we have that the background evolution can be written, in conformal time, as

$$\begin{aligned}
a(\eta) &= [-\frac{\eta}{\eta_1}]^{-\frac{1}{3}}, \qquad \eta \leq -\eta_1, \qquad\quad \Rightarrow\; p + \rho < 0\\
a(\eta) &= [\frac{2\eta_1 - \eta}{3\eta_1}]^{-1}, \qquad -\eta_1 < \eta < \eta_2, \quad \Rightarrow\; p + \rho = 0\\
a(\eta) &= \frac{(\eta + 2\eta_1 - \eta_1)\eta_1}{(2\eta_1 - \eta_2)^2}, \qquad \eta > \eta_2, \quad \Rightarrow\; p + \rho > 0.
\end{aligned} \tag{10}$$

The evolution equation of the proper amplitude of each polarization of the tensor fluctuations h_{ij} of a conformally flat background metric of Friedmann-Robertson-Walker type can be written, in conformal time and in Fourier space, as

$$\psi'' + [k^2 - \frac{a''}{a}]\psi = 0, \tag{11}$$

where for each polarization $\psi = ah$ (the prime denotes derivation with respect to conformal time).

The amplification induced by the background of Eq. (10) in the proper amplitude of the gravitational waves can be obtained from the solutions of Eq. (11) in the three temporal regions separated by η_1 and η_2:

$$\psi_I(k\eta) = \sqrt{\frac{\pi}{2}} e^{-i\frac{2\pi}{3}} \frac{1}{\sqrt{2k}} H^{(2)}_{5/6}(k\eta), \qquad \eta \leq -\eta_1,$$

$$\psi_{II}(k\eta) = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2k}}[e^{-i\pi}A_+(k)H^{(2)}_{3/2}(k\eta) + A_-(k)e^{i\pi}H^{(1)}_{3/2}(k\eta)], \qquad -\eta_1 < \eta < \eta_2,$$
$$\psi_{III}(k\eta) = \frac{1}{\sqrt{2k}}[B_+(k)e^{-ik\eta} + B_-(k)e^{ik\eta}], \qquad \eta > \eta_2 \tag{12}$$

where $H^{(1,2)}_\nu$ are the Hankel functions. By matching in $\eta = -\eta_1$ $\psi_{I,II,III}$ and $\psi'_{I,II,III}$ in $-\eta_1$ and η_2 we can determine the mixing coefficient $A_\pm(k)$, $B_\pm(k)$ whose frequency behavior will fix the spectral evolution of the energy density.

This is a very straightforward calculation [11] and the final result can be expressed in terms of the energy density of the relic gravitons at the present time and in critical units

$$\Omega_{\rm GW}(\omega, t_0) = \Omega_\gamma(t_0)s^2, \qquad \omega_2 < \omega < \omega_1$$
$$\Omega_{\rm GW}(\omega, t_0) = \Omega_\gamma(t_0)s^2\left(\frac{\omega}{\omega_2}\right)^{\frac{4}{3}}, \qquad \omega_{\rm dec} < \omega < \omega_2 \tag{13}$$

where $s \sim H_1/M_P$; $\Omega_\gamma \sim 2.6 \times 10^{-5}h_0^{-2}$ is the present fraction of critical energy density stored in radiation and $\omega = k/a = 2\pi\nu$.

Now our argument is very simple. The only free parameter in this model is given by the duration of the intermediate phase saturating the DOC, or, in other words by the ratio ω_1/ω_2. The overall normalization of the spectrum can be fixed by assuming, at ω_1, the maximal amplitude compatible with the nucleosynthesis bound $\Omega_{\rm GW}\lesssim 10^{-5}$. Taking into account that $\omega_1 \sim 10^{11}$ Hz we must conclude that the energy density of the relic gravitons at the scale of the interferometric detectors (i.e. $\omega_I \sim 0.1$ kHz) is given by

$$\Omega_{\rm GW} \sim 10^{-5}(\frac{\omega_I}{\omega_2})^{4/3}. \tag{14}$$

Therefore, if the duration of the intermediate phase lies in the range $|\eta_2/\eta_1| \sim 10^7$–$10^8$ we can conclude that the energy density of relic gravitons can be as large as 10^{-7}–10^{-6} in the frequency range probed by the interferometers without conflicting with other bounds coming from lower frequencies.

We showed that the existence of tilted spectra of relic gravitons can be connected, in the context of general relativity, with the violation of the DOC induced by bulk viscous stresses. In many respects what we presented in this paper are toy examples which are only suggestive of a possible dynamics. In the near future these speculations can have some phenomenological impact. Possible indications of the existence of tilted gravitons spectra at the frequency of the interferometric devices can put bounds on the possible violation of the DOC in the early Universe. If tilted gravitons spectra of the type discussed in the present paper will be detected, then the conclusion is twofold. Either the DOC was violated in the early Universe, or the theory of gravity underlying the correct model describing the early stages of the evolution of the Universe was not of Einstein-Hilbert type [12].

References

[1] L. P. Grishchuck, Zh. Eksp. Teor. Fiz. **67**,825 (1974) [Sov. Phys. JEPT 40, 409 (1975)]; L. P. Grishchuk and M. Solokhin, Phys. Rev. D **43**, 2566 (1991); L. P. Grishcuk and Y. V. Sidorov, Phys. Rev. D **42**, 3413 (1990); L. P. Grishchuk, Class. Quantum Grav. **10**, 2449 (1993).

[2] A. Guth. Phys. Rev. D 23, 347 (1981); A. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 122 (1982); A. Linde, Phys. Lett. B 129, 177 (1983); A. Linde, Phys. Rev. D **49**, 748 (1994).

[3] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, (Cambridge University Press, Cambridge, 1973).

[4] L. P. Grishchuk, *The Detectability of Relic (Squeezed) Gravitational Waves by Laser Interferometers*, gr-qc/9810055.

[5] L. P. Grishchuk, Phys. Rev. D **48**, 3513 (1993); *ibid.* **50**, 7154 (1994).

[6] J. D. Barrow, Nucl. Phys. **B310**, 743 (1988).

[7] P. C. W. Davies, Class. Quantum Grav. **4**, L225 (1987); Class. Quantum Grav. **5**, 1349 (1988).

[8] J. D. Barrow, Phys. Lett. **B187**, 12 (1987).

[9] S. Weinberg, *Gravitation and Cosmology*, (John Wiley, New York, 1972).

[10] V. N. Lukash and A. A. Starobinsky, *ibid.* **66**, 1515 (1974) [Sov. Phys. JETP **39**, 742 (1974)]; B. L. Hu and L. Parker, Phys. Rev. D **17**, 933 (1978); K. Jacobs, Astrophys. J. **153**, 661 (1968); K. Maeda, Phys. Rev. D **30**, 2482 (1984).

[11] M. Giovannini, Phys. Rev. D **58**, 083504 (1998); hep-ph/9903263.

[12] M. Gasperini and M. Giovannini, Phys. Lett. **B282**, 36 (1992); Phys. Rev.D **47**, 1519 (1993); M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Rev. D **48**, 439 (1993); M. Giovannini, Phys. Rev. D **55**, 595 (1997).